5.I: the ***PhilosophyIdeas*** overview of

Semantics of Logic

The game of chess is sometimes seen as a model of ancient warfare, with pawns standing for soldiers, and so on. This is an 'interpretation' of chess, but the game itself consists of objects (which could have any name), and rules for their behaviour. In logic, we say that there is an underlying '**syntax**' of entities and rules, to which we can (if we wish) add an interpretative '**semantics**'. The syntax can be implemented on a computer, but semantics may involve human understanding. Since we usually wish to reason about the world, semantics is normally interpreted in terms of what is '**true** (**T**)' or '**false** (**F**)', but it also includes theories of reference. Of course computers could do semantics, by assigning Ts and Fs (or 1s and 0s), but without some understanding there would be little point. Semantics is usually expressed in a metalanguage, since use of the language itself can lead to paradox (such as The Liar).

Propositional (or sentential) **Logic** (PL) concerns logical relationships between propositions (or sentences), and a proposition is defined as a complete assertion that is capable of being true or false. Hence the application of T or F to each proposition is the obvious first step in providing it with a semantics, and all that remains is to also interpret their relationships in terms of T or F. The relationships are the familiar logical terms, and their semantics is the truth table for each one. Thus 'and' (∧) delivers T if both propositions are true, 'or' (∨) if one of them is true, 'not' if the single proposition itself is false, and 'if…then' (→; 'material implication') as long as the left proposition is not true with the right proposition being false. Each line of a truth table can be thought of as one possible 'model' of the formula, built out of Ts and Fs, and if every line of the table delivers T then the proposition is a 'tautology'.

It is the Law of Excluded Middle which insists that each proposition must be T or F, but tricky propositions (involving vagueness, for example) may fall between the two. Classical logic ignores these tricky cases, but non-classical propositional logic tries to find a semantics for them, by assigning them something like 'U' (for 'undecided'). Kleene Logic is an example of a set of three-valued truth tables, produced by this approach. If some propositions fall between T and F, this is called a 'truth-value **gap**'. An even more alarming semantics might produce a '**glut**' of truth-values, meaning that some of the propositions are both T and F (as when you might answer 'well…yes and no' to a question).

Predicate Calculus (PC) deals with the ingredients of propositions, so the semantics is more complex. To PL we add quantifiers, variables, objects, predicates and relations. An 'atomic' sentence is a combination of an object and a predicate (perhaps written 'Fx', with a variable, or 'Fa', with an object). These are propositions, and can be assigned T or F, as before, and complex sentences can be built up, and T or F assigned to the complex sentence. However, there is scope for a much richer semantics. Just as we could assigns Ts and Fs to make single-line 'models' of formulae in a truth table, so we can specify a complex semantics for sets of PC sentences, using **Model Theory**.

Early semantic theories relied on 'concepts', but these proved troublesome, and set theory was then employed for the task. This semantics was '**extensional**', meaning all interpretations are in terms of objects; 'trilateral' and 'triangular' have the same extensional meaning, which is the set of all three-sided figures (and so the two can be substituted for one another), even though they are different concepts. The set theory semantics has one set of objects, and another of predicates (though the predicates are understood extensionally, as sets of objects, such as 'all the red things'). We then have an 'assignment' of these sets to the object variables (x,y,z..) and predicate variables (F,G,H…). We consider every possible permutation of these assignments (Fa, Gb etc), and a more complex semantics can add further sets for relations and functions, giving us a complete model of a set of formulae, with appropriate deliverances of T or F from each model. This semantic system has become the main tool for modern explorations of logic.

Once we have semantic models, we can introduce the idea of **semantic proof** (⊨), alongside deductive proof (⊢). Where the latter typically uses the rules of natural deduction to show what lines of reasoning do or do not lead to contradiction, the semantic approach follows the intuitive idea of refuting an argument by finding a counterexample. In model theory this would be an assignment of Ts which makes the whole model F. Thus the theory 'we met in London on Tuesday' is falsified if we assign T to 'I was in Rio all week' and 'you were in Beijing all week'. We can add background geographical information to the model to show why it comes out false.

A standard approach is to treat the semantics of a logical sentence as '**compositional**', meaning that the truth-value of complex statements depends entirely on the truth-values of their ingredients. Problem cases arise if the ingredients are not truth-functional (i.e. they don't guarantee T or F as output), and the weaker idea of '**satisfaction**' can be used; we say that 'x is F' is satisfied by 'a' if we can say 'a if F' (assigning names to variables, rather than objects to names). Thus we can assemble the semantics of a system like a jigsaw, without saying too much about what is actually true. A formal definition of '**truth**' can then be built up from this weaker idea of 'satisfaction'. The semantics of logic was a rather demoralised subject prior to the arrival of this definition (which gave an account of what 'T' actually means).

One issue for the semantics of logic is how to understand the idea of a **logical truth**. At first they were just taken to be what is simple and self-evident, but that rules out long and complex logical truths. They are sometimes said to be true only because of their logical language, and can thus survive all substitutions of the non-logical terms. This can be generalised as 'true by form', but another way to see them is as 'true in all models'; a logical truth is written as '⊨ φ', with no left-hand side, meaning φ is true with no premises at all. Logical truths are also understood as by-products of the rules of logic, making logical truth a more syntactic and less semantic notion.

Within the semantic systems expressed as models, there are separate issues concerning the meanings of object-terms and predicate-terms, the first concerning the problem of **reference**, and the second the metaphysics of **properties**. These issues encounters the problem that we can't normally relate symbolic logic directly to the world (which would make the semantics a lot simpler!), but the connection has to be specified in **ordinary language**. Semantics for logic is very precise, but the semantics of natural languages is deeply perplexing, and studies theories of meaning, rather than relations involving T and F.